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PROBABILITIES IN THE (k,ℓ) HOOK

BY

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ABSTRACT

We consider particular (k,ℓ)-hook probability measures on the space of the infinite standard Young tableaux, and calculate the probability that the entry at the $(1,2)$ cell is odd. As n goes to infinity, this, approximately, is the corresponding probability on tableaux of size n in the (k,ℓ) hook. In few cases of small k and ℓ we find exact formulas for the corresponding numbers of such standard tableaux.

1. Introduction

Let f^λ denote the number of standard tableaux of shape λ. Given two partitions $\mu \subset \lambda$, let $f^{\lambda/\mu}$ denote the number of standard tableaux of the corresponding skew shape λ/μ.

The motivation for the present work is the following phenomena proved in [11], see also [6]: Consider the natural — i.e. Plancherel — probability on standard tableaux. Then the probability that in a large random standard tableau the $(1,2)$ entry is odd, is approximately $1/e$, where $e = 2.718\ldots$. In the present paper we study the (k,ℓ)-hook analogue of that phenomena.

More generally, we study the probability — in the (k,ℓ) hook — that the (i,j) entry in a standard tableau is of a given value m. Recall that $H(k,\ell;n)$ are the partitions of n in the (k,ℓ) hook: $H(k,\ell;n) = \{\lambda = (\lambda_1,\lambda_2,\ldots) \vdash n \mid \lambda_{k+1} \le \ell\}$, and $H(k,\ell) = \bigcup_{n=1}^{\infty} H(k,\ell;n)$, called the (k,ℓ) hook. If T is a tableau of shape

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$\lambda \in H(k,\ell)$, we say that T is in the (k,ℓ) hook. Given a tableau T and a cell (i,j), $T(i,j)$ denotes the (i,j) entry in T.

Thus, we study the following problem: For fixed $k,\ell,(i,j),m$ and n, what is the probability that in a random standard tableau T_λ of shape $\lambda \in H(k,\ell;n)$, $T_\lambda(i,j)=m$. This probability $P_{(k,\ell)}(T(i,j)=m)$ is the ratio a/b where b is the total number of standard tableaux T_λ of shape λ, $\lambda \in H(k,\ell;n)$, while a is the number of such tableaux T_λ where $T_\lambda(i,j)=m$.

The problem of computing these probabilities precisely seems rather hard for each n, and we first solve it asymptotically. Theorem 3.3 below, which is a special case of [12, Theorem 4.1.a], can be interpreted as giving the limit such probability, as n goes to infinity. Note that [12, Theorem 4.1] is a consequence of the Vershik–Kerov theory of the S_∞ characters [14], [16], [15]. For a brief summary of that theory, see [12, Section 2], and also [8]. In the second part, from Section 6 on, we compute precisely, for each n, several special cases of these probabilities.

In this paper we mostly restrict our discussion to the case $(i,j)=(1,2)$. Theorem 4.1 gives the corresponding probability in that case. Summing over all odd m, Proposition 5.1 gives a formula for the corresponding probability that the $(1,2)$ entry, in such random tableau, is odd. The hook-Schur functions $HS_\lambda(x;y)$ [1] play an important role in deducing that formula.

For standard tableaux T with shapes in $H(k,\ell;n)$, the probability that $T(1,2)$ is odd is the ratio $S_{odd}(k,\ell;n)/S(k,\ell;n)$. Here

$$STH(k,\ell;n)=\{T_\lambda \text{ is standard of shape } \lambda : \lambda \in H(k,\ell;n)\},$$
$$S(k,\ell;n)=|STH(k,\ell;n)|=\sum_{\lambda\in H(k,\ell;n)} f^\lambda$$

and

$$S_{odd}(k,\ell;n)=|\{T_\lambda \in STH(k,\ell;n): T_\lambda(1,2) \text{ is odd}\}|=\sum_{\lambda\in H(k,\ell;n)} h^\lambda,$$

where h^λ is the number of standard tableaux T_λ of shape λ with $T_\lambda(1,2)$ being odd, see [11] for a representation-theory interpretation of the numbers h^λ. Thus, Proposition 5.1 yields the limit value of $S_{odd}(k,\ell;n)/S(k,\ell;n)$ as n goes to infinity.

It is natural to look for efficient closed formulas for the combinatorial sums $S(k,\ell;n)$ and $S_{odd}(k,\ell;n)$. By efficient closed formula we mean one that does not involve double, or more, summations. In Sections 5–10 we deduce such formulas in few cases where k and ℓ are small. Closed formulas for the sums $S(k,0;n) = S(0,k;n)$ are known for $k \le 5$, see [5], [10], [13, Ex.7.16]. Here we give such formulas for $S(1,1;n)$ and for $S(2,1;n) = S(1,2;n)$. We also give such formulas for the sums $S_{odd}(k,\ell;n)$ for $(k,\ell) \in \{(2,0),\ (0,2),\ (1,1),\ (2,1)\}$, see the next section for a description of these cases.

The On-Line Encyclopedia for Integer Sequences was useful in the study of some of these cases. Thanks are also due to G. Olshansky for some very helpful suggestions, see, in particular, Theorem 3.6.

2. The main results

In Sections 3 and 5, we study the probability $P_{(k,\ell)}(T(i,j) = m)$, see Definition 3.1 below.

Theorem 3.6 shows that the probability $P_{(k,\ell)}(T(i,j) = m)$ is the limit of a sequence of probabilities on certain finite sets of finite tableaux.

Applying [12, Theorem 4.1.a] and some properties of hook-Schur functions [1], we prove

Theorem 2.1 (Theorem 4.1): *Let* $(i,j) = (1,2)$, *then*

$$P_{(k,\ell)}(T(1,2) = m) = \left(\frac{1}{k+\ell}\right)^m \cdot \left[(m-1)\binom{\ell+m-2}{\ell-2} + \sum_{r=1}^{m-1} \frac{r(k+\ell+1)+\ell}{r+1} \cdot \binom{k}{r} \cdot \binom{\ell+m-2-r}{\ell-1}\right].$$

Summing over all odd numbers, it implies

Proposition 2.2 (Proposition 5.1):

$$P_{(k,\ell)}(T(1,2) \text{ is odd}) = \sum_{t=1}^{\infty} \left(\frac{1}{k+\ell}\right)^{2t+1} \left[2t\binom{\ell+2t-1}{\ell-2} + \sum_{r=1}^{2t} \frac{r(k+\ell+1)+\ell}{r+1} \cdot \binom{k}{r} \cdot \binom{\ell-1+2t-r}{\ell-1}\right].$$

This equation is applied, in Section 5, to calculate few cases of $P_{(k,\ell)}(T(1,2)$ is odd) with small k and ℓ.

In Sections 5–10, we deduce closed formulas for the combinatorial sums $S(k,\ell;n)$ and $S_{odd}(k,\ell;n)$ for some low cases of (k,ℓ). Consider first $S(k,\ell;n)$. Closed formulas for the sums $S(k,0;n)=S(0,k;n)$ are known for $k\le 5$ [13, Exercise 7.16]. Turn now to the cases $k,\ell\neq 0$. The hook-formula for f^λ easily implies that $S(1,1;n)=2^{n-1}$. In Section 8, we prove an explicit formula for the sums $S(2,1;n)=S(1,2;n)$. As far as we know, so far there are no known effective closed formulas for further cases.

Turn now to $S_{odd}(k,\ell;n)$ which, in general, is not equal to $S_{odd}(\ell,k;n)$. In Section 7 we study $S_{odd}(2,0;n)$ and $S_{odd}(0,2;n)$. A closed formula for $S_{odd}(2,0;n)$ follows by observing that $S_{odd}(2,0;n)=S(2,0;n-2)$. The Catalan numbers C_m determine $S(2,0;n)$, since $S(2,0;2m)=2S(2,0;2m-1)$ and $S(2,0;2m+1)=2S(2,0;2m)-C_m$. Since $S_{odd}(2,0;n)=S(2,0;n-2)$, it follows that $S_{odd}(2,0;n)$ is also determined by the Catalan numbers. It is well-known that $C_m=f^{(m,m)}=f^{(2^m)}$. Recall the Fine numbers F_m and their relation to the Catalan numbers: $C_m=2F_m+F_{m-1}$, see, for example, [2]. Lemma 7.2 shows that $h^{(2^m)}=F_m$, the m-th Fine number. As far as we know, this gives a new interpretation of the Fine numbers in terms of certain standard tableaux, see Remark 7.3. Proposition 7.1 gives $S_{odd}(0,2;n)$ in terms of the Fine numbers as follows:

$$S_{odd}(0,2;2m-1)=\frac{1}{9}\left[\binom{2m+1}{m}+2F_m-3\right]\quad\text{and}$$
$$S_{odd}(0,2;2m)=\frac{2}{9}\left[\binom{2m+1}{m}+2F_m-3\right].$$

The details are given in Section 7 and in the Appendix.

A closed formula for the sum $S_{odd}(1,1;n)$ is given by Equation (17). A closed formula for $S_{odd}(2,1;n)$ is given in Section 9, where we prove that $S_{odd}(2,1;n+1)=S(2,1;n)-1$. The proof of that surprising relation involves verifying a non-trivial binomial identity, see Equation (20) below. We are thankful to D. Zeilberger for verifying that identity by the WZ method. Together with the closed formula for $S(2,1;n)$ in Section 8, this yields a closed formula for $S_{odd}(2,1;n)$. Finally, for $S_{odd}(3,0;n)$, numerical evidence suggest the following intriguing conjecture: $S_{odd}(3,0;n)=S(3,0;n-1)-S(3,0;n-3)$. It is natural to look for bijective proofs for the above identities, see Remark 10.4. So far, only the proof of the identity $S_{odd}(2,0;n)=S(2,0;n-2)$ is bijective.

3. Asymptotic probabilities in the (k, ℓ) hook

We consider the Vershik-Kerov ergodic measures $M(\alpha;\beta;\gamma)$ on the set Tab of the infinite standard tableaux, and the corresponding extended Schur functions $\tilde{S}_\nu(\alpha;\beta;\gamma)$ (see [12]). Here α and β are infinite sequences of descending non-negative real numbers $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$; $\beta_1 \geq \beta_2 \geq \cdots \geq 0$, satisfying $\sum_{i=1}^{\infty}(\alpha_i + \beta_i) \leq 1$, and $\gamma = 1 - \sum_{i=1}^{\infty}(\alpha_i + \beta_i)$. Then $M(\alpha;\beta;\gamma)$ is the corresponding ergodic measure on the space Tab. Also, $\tilde{S}_\nu(\alpha;\beta;\gamma)$ are the corresponding Vershik–Kerov extended Schur functions.

Let $\alpha_1 \geq \cdots \geq \alpha_k \geq 0, \beta_1 \geq \cdots \geq \beta_\ell \geq 0$, $\alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_\ell = 1$, $\alpha_q = 0$ for all $q > k$, $\beta_q = 0$ for all $q > \ell$ and $\gamma = 0$. Then $\tilde{S}_\nu(\alpha;\beta;\gamma) = HS_\nu(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_\ell)$ is the hook (or "super") Schur function, see [1], [14], [16], [15].

Definition 3.1:

1. Let (i,j) be a fixed cell, let $m \in \mathbb{N}$ and let $\mathcal{P}_{M(\alpha;\beta;\gamma)}(T(i,j) = m)$ denote the probability, with respect to the measure $M(\alpha;\beta;\gamma)$, that the (i,j) entry in a random standard tableau equals m, with respect to the measure $M(\alpha;\beta;\gamma)$. Further, make the choice
$$\alpha_1 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_\ell = \frac{1}{k+\ell}.$$
With these α, β (and $\gamma = 0$) we define
$$P_{(k,\ell)}(T(i,j) = m) \;=\; \mathcal{P}_{M(\alpha;\beta;0)}(T(i,j) = m).$$
2. Let T be an infinite standard tableau and let $shape_m(T)$ denote the shape $\nu \vdash m$ formed by the entries $1, \ldots, m$ in T. Similarly, when $n \geq m$ and T is a finite standard tableau with $|T| = n$, again $shape_m(T)$ denotes the shape $\nu \vdash m$ formed by the entries $1, \ldots, m$ in T. With the above α, β and $\gamma = 0$, define
$$P_{(k,\ell)}(shape_m(T) = \mu) = \mathcal{P}_{M(\alpha;\beta;0)}(shape_m(T) = \mu).$$

The definitions of the subsets
$$\{T : T(i,j) = m\}, \{T : shape_m(T) = \mu\} \subseteq Tab$$
are obvious. These subsets are related by the following observation, which was also applied in the proof of [12, Theorem 4.1.a].

LEMMA 3.2:

$$\begin{aligned}&\{T : T(i,j) = m\}\\ &\quad = \bigcup_{\mu\in H'(i-1,j-1;m-1)} \{T : \mathit{shape}_{m-1}(T) = \mu \ \text{ and } \mathit{shape}_m(T) = \mu^+(i,j)\},\end{aligned}$$

a disjoint union. Here $H'(i-1, j-1; m-1)$ *are the partitions* $\mu \in H(i-1, j-1; m-1)$ *such that when adding the cell* (i,j) *to* μ*, the result, which is denoted* $\mu^+(i,j)$*, is a partition; namely,* $\mu_{i-1} \geq j$ *and* $\mu'_{j-1} \geq i$*. In particular, for any probability measure* $\mathcal{P}_M$ *on* Tab*,*

$$\begin{aligned}(1)\quad &\mathcal{P}_M(T(i,j) = m)\\ &\quad = \sum_{\mu\in H'(i-1,j-1;m-1)} \mathcal{P}_M(\mathit{shape}_{m-1}(T) = \mu \text{ and } \mathit{shape}_m(T) = \mu^+(i,j)).\end{aligned}$$

THEOREM 3.3:

$$P_{(k,\ell)}(T(i,j) = m) = \left(\frac{1}{k+\ell}\right)^m \cdot \sum_{\mu\in H'(i-1,j-1;m-1)} f^\mu \cdot HS_{\mu^+(i,j)}(1^k; 1^\ell).$$

Proof. This is a special case of [12, Theorem 4.1.a]

$$(2)\qquad \mathcal{P}_{M(\alpha;\beta;\gamma)}(T(i,j) = m) = \sum_{\mu\in H'(i-1,j-1;m-1)} f^\mu \cdot \tilde{S}_{\mu^+(i,j)}(\alpha;\beta;\gamma).$$

Recall that $\tilde{S}_\nu(\alpha;\beta;\gamma) = HS_\nu(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_\ell)$ and note that for any partition ν, $HS_\nu(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_\ell)$ is homogeneous of degree $|\nu|$. It follows that

$$\begin{aligned}&HS_\nu(\underbrace{1/(k+\ell),\ldots,1/(k+\ell)}_{k};\underbrace{1/(k+\ell),\ldots,1/(k+\ell)}_{\ell})\\ &\qquad\qquad = \left(\frac{1}{k+\ell}\right)^{|\nu|} \cdot HS_\nu(1^k;1^\ell),\end{aligned}$$

which completes the proof. ∎

Theorem 3.6 below shows that the probability $P_{(k,\ell)}(T(i,j) = m)$ is the limit of a sequence of probabilities on certain finite sets of finite tableaux.

Definition 3.4: 1. Let ST_λ denote the standard tableaux of shape λ and denote

$$STH(k,\ell;n) = \bigcup_{\lambda\in H(k,\ell;n)} ST_\lambda \quad \text{so} \quad |STH(k,\ell;n)| = \sum_{\lambda\in H(k,\ell;n)} f^\lambda. \tag{3}$$

2. Let $ST_\lambda(T(i,j) = m) = \{T \in ST_\lambda : T(i,j) = m\}$ and $h^\lambda_{(i,j);m} = |ST_\lambda(T(i,j) = m)|$. Also let

$$STH(k,\ell;n)(T(i,j) = m) = \bigcup_{\lambda\in H(k,\ell;n)} ST_\lambda(T(i,j) = m) \quad \text{so}$$

$$|STH(k,\ell;n)(T(i,j) = m)| = \sum_{\lambda\in H(k,\ell;n)} h^\lambda_{(i,j);m}. \tag{4}$$

3. Let

$$ST_\lambda(shape_m(T) = \mu) = \{T \in ST_\lambda \mid shape_m(T) = \mu)\} \quad \text{and}$$
$$h^\lambda_\mu = |ST_\lambda(shape_m(T) = \mu)|.$$

Also let

$$STH(k,\ell;n)(shape_m(T) = \mu) = \bigcup_{\lambda\in H(k,\ell;n)} ST_\lambda(shape_m(T) = \mu) \quad \text{so}$$

$$|STH(k,\ell;n)(shape_m(T) = \mu)| = \sum_{\lambda\in H(k,\ell;n)} h^\lambda_\mu. \tag{5}$$

Similar to Lemma 3.2, also in the case of finite tableaux we obviously have the following

LEMMA 3.5: *Let $m \le n$, then*

$$STH(k,\ell;n)(T(i,j) = m)$$
$$= \bigcup_{\mu\in H'(i-1,j-1;m-1)} STH(k,\ell;n)(shape_{m-1}(T)=\mu \text{ and } shape_m(T)=\mu^+(i,j)),$$

a disjoint union.

THEOREM 3.6:

$$P_{(k,\ell)}(shape_m(T) = \mu) = \lim_{n\to\infty} \frac{|\{STH(k,\ell;n)(shape_m(T) = \mu)\}|}{|STH(k,\ell;n)|} \tag{6}$$

and

$$P_{(k,\ell)}(T(i,j) = m) = \lim_{n\to\infty} \frac{|\{STH(k,\ell;n)(T(i,j) = m)\}|}{|STH(k,\ell;n)|}. \tag{7}$$

Proof. (G. Olshanski) Note first that by (1) and by Lemma 3.5, (6) implies (7), so we prove (6). Let $P^{(n)}$ denote the uniform measure on the finite set $STH(k,\ell;n)$, so that the weights of all tableaux in $STH(k,\ell;n)$ are the same and equal to $|STH(k,\ell;n)|^{-1}$. The measure $P^{(n)}$ induces a probability measure, say $\bar{P}^{(n)}$, on the set of Young diagrams λ with n cells, contained in the (k,ℓ)-hook: the weight $\bar{P}^{(n)}(\lambda)$ is proportional to f^λ.

Let $T^{(n)}$ denote a random tableau from $STH(k,\ell;n)$. Let $m<n$. The probability (with respect to $P^{(n)}$) that $shape_m(T^{(n)})$ coincides with a given Young diagram μ with m cells is equal to

$$P^{(n)}(shape_m(T^{(n)})=\mu)=|STH(k,\ell;n)|^{-1}\sum_{\lambda\in H(k,\ell;n)} f^\mu f^{\lambda/\mu}.$$

The above probability can be written as

$$P^{(n)}(shape_m(T^{(n)})=\mu)=\sum_{\lambda\in H(k,\ell;n)}\frac{f^\mu f^{\lambda/\mu}}{f^\lambda}\cdot\bar{P}^{(n)}(\lambda). \tag{8}$$

Next, using formula (0.3) of [9], we express the ratio $\frac{f^{\lambda/\mu}}{f^\lambda}$ through the Frobenius–Schur function Fs_μ evaluated at the modified Frobenius coordinates $(x,y)=(x(\lambda),y(\lambda))$ of λ (in our situation, the number of (non-zero) coordinates is $d\le\max(k,\ell)$):

$$\frac{f^{\lambda/\mu}}{f^\lambda}=\frac{Fs_\mu(x,y)}{n(n-1)\cdots(n-m+1)}=s_\mu(\tfrac{1}{n}x,\tfrac{1}{n}y)+O(\tfrac{1}{n}). \tag{9}$$

Here s_μ is the hook-Schur function $s_\mu=HS_\mu$.

We now show that as $n\to\infty$, the measure $\bar{P}^{(n)}$, viewed as a measure on the normalized modified Frobenius coordinates $(\frac{1}{n}x,\frac{1}{n}y)$, concentrates near the point (α,β) where

$$\alpha_1=\cdots=\alpha_k=\beta_1=\cdots=\beta_\ell=\frac{1}{k+\ell}\quad(\text{and }\alpha_{k+1}=\beta_{\ell+1}=0).$$

Together with (8) and (9) this implies that (8) tends to $f^\mu s_\mu(\alpha,\beta)$, that is, to the probability corresponding to the measure $M_{\alpha,\beta}$. This, in turn, implies the relation (6).

Finally, we prove the above statement about $\bar{P}^{(n)}$. Given $\lambda\in H(k,\ell;n)$, write

$$\lambda_i=\frac{n}{k+\ell}+c_i(\lambda)\sqrt{n},\ 1\le i\le k,\quad\text{and}\quad\lambda'_j=\frac{n}{k+\ell}+c'_j(\lambda)\sqrt{n},\ 1\le j\le\ell.$$

Let $0 < a$ and denote

$$H(k,\ell;n;a) = \{\lambda \in H(k,\ell;n) : |c_i(\lambda)|, |c'_j(\lambda)| < a\},$$

with corresponding

$$STH(k,\ell;n;a) = \{T \in STH(k,\ell;n) : shape(T) \in H(k,\ell;n;a)\}$$

and

$$STH(k,\ell;n;a)(T(i,j)=m) \\ = \{T \in STH(k,\ell;n)(T(i,j)=m) : shape(T) \in H(k,\ell;n;a)\}$$

Note that for a large n, if $\lambda \in H(k,\ell;n;a)$ then $\lambda_1,\dots,\lambda_k,\lambda'_1,\dots,\lambda'_\ell \approx n/(k+\ell)$, namely, λ is nearly (k,ℓ) rectangular. Now, it follows from [1, Section 7] that given $0 < \varepsilon$, there is $0 < a$ such that when $n \to \infty$,

$$1-\varepsilon < \frac{|STH(k,\ell;n;a)|}{|STH(k,\ell;n)|} < 1 \qquad \text{namely,} \qquad \frac{|STH(k,\ell;n;a)|}{|STH(k,\ell;n)|} \approx 1.$$

Of course, as ε becomes smaller, we need to take larger a. Thus, when $n \to \infty$,

$$\frac{|\{STH(k,\ell;n)(T(i,j)=m)\}|}{|STH(k,\ell;n)|} \approx \frac{|\{STH(k,\ell;n;a)(T(i,j)=m)\}|}{|STH(k,\ell;n;a)|}.$$

Finally, for fixed a and with $n \to \infty$, if $\lambda \in H(k,\ell;n,a)$, then $\lambda_1,\dots,\lambda_k$, $\lambda_1,\dots,\lambda_\ell \approx n/(k+\ell)$ as was claimed. The proof is now complete. ∎

Remark 3.7: We just saw that for large n, most tableaux in $STH(k,\ell;n)$ have shapes $\lambda \in H(k,\ell;n)$ where $\lambda_1,\dots,\lambda_k$ and $\lambda'_1,\dots,\lambda'_\ell$ are all close to $n/(k+\ell)$. Thus, for such "typical" λ, h^λ/f^λ is close to $P_{(k,\ell)}(T(i,j)=m)$, namely,

$$h^{\lambda}_{(i,j);m} \approx P_{(k,\ell)}(T(i,j)=m)\cdot f^\lambda. \tag{10}$$

4. Some special cases

In the rest of this paper we calculate the probabilities $P_{(k,\ell)}(T(i,j)=m)$ as the limit $n \to \infty$ of the finite probabilities $P^{(n)}$.

4.1. THE CASE $(i,j)=(1,2)$. In this paper we restrict our discussion to the case $(i,j)=(1,2)$. By conjugation, this includes also the case $(i,j)=(2,1)$. As a corollary of Theorem 3.3 we have the following formula.

THEOREM 4.1: *Let* $(i,j)=(1,2)$, *then*

$$\begin{aligned}(11)\quad & P_{(k,\ell)}(T(1,2)=m)\\ &=\left(\frac{1}{k+\ell}\right)^m\cdot HS_{(2,1^{m-2})}(1^k;1^\ell)\\ &=\left(\frac{1}{k+\ell}\right)^m\cdot\Bigg[(m-1)\binom{\ell+m-2}{\ell-2}\\ &\qquad+\sum_{r=1}^{m-1}\frac{r(k+\ell+1)+\ell}{r+1}\cdot\binom{k}{r}\cdot\binom{\ell+m-2-r}{\ell-1}\Bigg].\end{aligned}$$

Before proving the theorem we observe the following corollary.

COROLLARY 4.2: 1.

$$P_{(k,0)}(T(1,2)=m)=\frac{1}{k^m}\cdot(m-1)\cdot\binom{k+1}{m}.$$

2.

$$P_{(0,\ell)}(T(1,2)=m)=\frac{1}{\ell^m}\cdot(m-1)\cdot\binom{\ell+m-2}{\ell-2}.$$

Proof. Note that when $\ell=0$ (or even $\ell=1$), the term $(m-1)\binom{\ell+m-2}{\ell-2}$ in (11) is zero. Also when $\ell=0$, the term $\binom{k}{r}\cdot\binom{\ell+m-2-r}{\ell-1}$ is zero unless $r=m-1$, at which case it is 1. This implies part 1. If $k=0$ then $\binom{k}{r}=0$ (since $r\geq 1$), which implies part 2. ∎

A simpler direct proof of this corollary is given below, see Remark 4.4.

We now prove Theorem 4.1.

Proof. To prove the theorem, note first that $H'(0,1;m-1)$ has the single element $\mu=(1^{m-1})$, with $f^\mu=1$ and $\mu^+(1,2)=(2,1^{m-2})$. By Theorem 3.3 this implies the first equality

$$P_{(k,\ell)}(T(1,2)=m)=\left(\frac{1}{k+\ell}\right)^m\cdot HS_{(2,1^{m-2})}(1^k;1^\ell).$$

The proof of Theorem 4.1 now follows from the following lemma on Hook–Schur functions. ∎

Lemma 4.3:

$$HS_{(2,1^{m-2})}(1^k,1^\ell) = (m-1)\binom{\ell+m-2}{\ell-2} + \sum_{r=1}^{m-1} \frac{r(k+\ell+1)+\ell}{r+1}\cdot\binom{k}{r}\cdot\binom{\ell+m-2-r}{\ell-1}.$$

Proof. By [1],

$$HS_\lambda(x_1,\ldots,x_k;y_1,\ldots,y_\ell) = \sum_{\phi\le\mu\le\lambda} S_\mu(x_1,\ldots,x_k)\cdot S_{(\lambda/\mu)'}(y_1,\ldots,y_\ell),$$

and here $\lambda = (2,1^{m-2})$. Let $\phi < \mu < (2,1^{m-2})$, then either $\mu = (1^r)$ with $0 \le r \le m-1$, or $\mu = (2,1^u)$ with $0 \le u \le m-2$.

$\boldsymbol{\mu = (1^r)}$. If $r = 0$, then $\mu = \phi$, $((2,1^{m-2})/\phi)' = (2,1^{m-2})' = (m-1,1)$, and we have the corresponding summand

$$S_0(1^k)\cdot S_{(m-1,1)}(1^\ell) = 1\cdot S_{(m-1,1)}(1^\ell) = 2t\binom{\ell+m-2}{\ell-2}$$

(see [7, page 45, Example 4], for a formula for $S_\lambda(1^k)$). When $1\le r\le m-1$, $(2,1^{m-2})/(1^r)$ has two components, (1) and (1^{m-1-r}), so $((2,1^{m-2})/(1^r))'$ has the components, (1) and $(m-1-r)$, which yield the summand

$$S_{(1^r)}(x_1,\ldots,x_k)\cdot S_{(1)}(y_1,\ldots,y_\ell)\cdot S_{(m-1-r)}(y_1,\ldots,y_\ell).$$

Substitute all $x_i = y_j = 1$. Since for any q, $S_q(1^\ell) = \binom{\ell+q-1}{\ell-1}$, and since $S_{(1^r)}(1^k) = \binom{k}{r}$, the corresponding summand for $1\le r\le m-1$ is

$$\binom{k}{r}\cdot\ell\cdot\binom{\ell-1+m-1-r}{\ell-1}.$$

$\boldsymbol{\mu = (2,1^u)}$ then $(2,1^{m-2})/(2,1^u) = (1^{m-2-u})$, which yields the summand

$$S_{(2,1^u)}(x_1,\ldots,x_k)\cdot S_{(m-2-u)}(y_1,\ldots,y_\ell),$$

and after substituting all $x_i = y_j = 1$ it contributes

$$S_{(2,1^u)}(1^k)\cdot S_{(m-2-u)}(1^\ell) = (u+1)\binom{k+1}{u+2}\cdot\binom{\ell+m-3-u}{\ell-1}.$$

Deduce that

$$
\begin{aligned}
&HS_{(2,1^{m-2})}(1^k,1^\ell)\\
&\quad=(m-1)\binom{\ell+m-2}{\ell-2}+\sum_{r=1}^{m-1}\binom{k}{r}\cdot\ell\cdot\binom{\ell+m-2-r}{\ell-1}\\
&\qquad+\sum_{u=0}^{m-2}(u+1)\cdot\binom{k+1}{u+2}\cdot\binom{\ell+m-3-u}{\ell-1}\\
&\quad=(m-1)\binom{\ell+m-2}{\ell-2}+\sum_{r=1}^{m-1}\binom{k}{r}\cdot\ell\cdot\binom{\ell+m-2-r}{\ell-1}\\
&\qquad+\sum_{r=1}^{m-1}r\cdot\binom{k+1}{r+1}\cdot\binom{\ell+m-2-r}{\ell-1}\\
&\quad=(m-1)\binom{\ell+m-2}{\ell-2}\\
&\qquad+\sum_{r=1}^{m-1}\frac{r(k+\ell+1)+\ell}{r+1}\cdot\binom{k}{r}\cdot\binom{\ell+m-2-r}{\ell-1}. \quad \blacksquare
\end{aligned}
$$

Remark 4.4: A direct proof of part 1 of Corollary 4.2 follows by similar (and simpler) arguments, from the equality

$$S_{(2,1^{m-2})}(\underbrace{1,\ldots,1}_{k})=(m-1)\binom{k+1}{m}.$$

Similarly, a direct proof of part 2 of Corollary 4.2 follows from

$$S_{(2t,1)}(\underbrace{1,\ldots,1}_{\ell})=\frac{(\ell-1)\cdot\ell\cdot(\ell+1)\cdots(\ell+2t-1)}{(2t-1)!\cdot(2t+1)}=2t\binom{\ell+2t-1}{\ell-2}.$$

5. The probability that $T(1,2)$ is odd

We study here the probability $P_{(k,\ell)}(T(1,2)$ is odd). Note that if T is a standard tableau of shape λ and $\lambda_1\geq 2$, then $T(1,2)\geq 2$. Thus

$$P_{(k,\ell)}(T(1,2)\text{ is odd})=\sum_{t=1}^{\infty}P_{(k,\ell)}(T(1,2)=2t+1),$$

and Theorem 4.1 implies the following formula (12) for calculating $P_{(k,\ell)}(T(1,2)$ is odd).

PROPOSITION 5.1:

$$(12)\quad P_{(k,\ell)}(T(1,2) \text{ is odd}) = \sum_{t=1}^{\infty}\left(\frac{1}{k+\ell}\right)^{2t+1}\left[2t\binom{\ell+2t-1}{\ell-2} + \sum_{r=1}^{2t}\frac{r(k+\ell+1)+\ell}{r+1}\cdot\binom{k}{r}\cdot\binom{\ell-1+2t-r}{\ell-1}\right].$$

In particular

$$(13)\qquad P_{(k,0)}(T(1,2) \text{ is odd}) = \sum_{t=1}^{[k/2]}\frac{1}{k^{2t+1}}\cdot 2t\binom{k+1}{2t+1}$$

and

$$(14)\qquad P_{(0,\ell)}(T(1,2) \text{ is odd}) = \sum_{t=1}^{\infty}\frac{1}{\ell^{2t+1}}\cdot 2t\binom{\ell+2t-1}{\ell-2}.$$

Definition 5.2: Let $\lambda \vdash n$. We denote by h^λ the number of standard tableaux of shape λ with the $(1,2)$-entry being odd. Thus $h^\lambda = \sum_{t\geq 1} h^\lambda_{(1,2);2t+1}$, see Definition 3.4.

Remark 5.3: It follows from (10) that h^λ/f^λ is close to $P_{(k,\ell)}(T(1,2) \text{ is odd})$, namely

$$(15)\qquad h^\lambda \approx P_{(k,\ell)}(T(1,2) \text{ is ddd})\cdot f^\lambda,$$

provided λ is typical in the (k,ℓ) hook.

5.1. SOME CASES OF $\ell = 0$.

Example 5.4: $\boldsymbol{k=2}$: Here the summands of (13) with $t>1$ equal 0, so $t=1$ and $P_{(2,0)}(T(1,2) \text{ is odd}) = \frac{1}{2^3}\cdot\frac{1\cdot 2\cdot 3}{1\cdot 3} = \frac{1}{4}$.

$\boldsymbol{k=3}$: Again $t=1$ and $P_{(3,0)}(T(1,2) \text{ is odd}) = 8/27$.

Similarly $P_{(4,0)}(T(1,2) \text{ is odd}) = 81/4^4$, $P_{(5,0)}(T(1,2) \text{ is odd}) = 1024/5^5$ and $P_{(6,0)}(T(1,2) \text{ is odd}) = 15625/6^6 = 2.986^{-1}$.

As k becomes larger, $P_{(k,0)}(T(1,2) \text{ is odd})$ tends to $e^{-1} = 2.7182818^{-1}$.

5.2. SOME CASES OF $k = 0$.

Example 5.5: $\boldsymbol{\ell = 2}$. By (14)

$$P_{(0,2)}(T(1,2) \text{ is odd}) = \sum_{t=1}^{\infty} \frac{1}{2^{2t+1}} \cdot \frac{(2t+1)!}{(2t-1)! \cdot (2t+1)} = \sum_{t=1}^{\infty} \frac{t}{4^t} = \frac{4}{9}$$
$$= 0.4444444\ldots.$$

(The last sum is evaluated as follows: let $f(x) = \sum_{t=1}^{\infty} x^t/4^t = x/(4-x)$. Now differentiate $f(x)$ and evaluate $f'(x)$ at $x = 1$: $f'(1) = 4/9$.)

$\boldsymbol{\ell = 3}$. Similarly,

$$P_{(0,3)}(T(1,2) \text{ is odd}) = \sum_{t=1}^{\infty} \frac{1}{3^{2t+1}} \cdot \frac{(2t+2)!}{(2t-1)! \cdot (2t+1)} = \frac{4}{3} \cdot \sum_{t=1}^{\infty} \frac{t(t+1)}{9^t} = \frac{27}{64}$$
$$= 0.421875\ldots.$$

(Define $f(x) = \frac{4}{3} \sum_{t \geq 1} \frac{x^{t+1}}{3^{2t}} = \frac{4}{3} \cdot \frac{x^2}{9-x}$, then evaluate the second derivative of $f(x)$ at $x = 1$.)

$\boldsymbol{\ell = 4}$. By similar arguments, $P_{(0,4)}(T(1,2) \text{ is odd}) = 0.4096$

5.3. Some (k, ℓ) cases. Here we apply (12).

Example 5.6: $\boldsymbol{(k, \ell) = (1, 1)}$. Here

$$P_{(1,1)}(T(1,2) \text{ is odd}) = \sum_{t \geq 1} \left(\frac{1}{2}\right)^{2t+1} \cdot \frac{4}{2} = \sum_{t \geq 1} \left(\frac{1}{4}\right)^{t} = \frac{1}{3}.$$

$\boldsymbol{(k, \ell) = (2, 1)}$. In (12) $r = 1, 2$ so

$$P_{(2,1)}(T(1,2) \text{ is odd}) = \sum_{t \geq 1} \left(\frac{1}{3}\right)^{2t+1} \cdot \left(\frac{5}{2}\binom{2}{1} + \frac{9}{3}\binom{2}{2}\right) = \frac{1}{3}.$$

$\boldsymbol{(k, \ell) = (1, 2)}$. Here

$$P_{(1,2)}(T(1,2) \text{ is odd}) = \frac{8}{3} \sum_{t \geq 1} \frac{t}{9^t} = \frac{3}{8}.$$

$\boldsymbol{(k, \ell) = (3, 1)}$. In (12), if $t = 1$, then $r = 1, 2$, and if $t \geq 2$ then $r = 1, 2, 3$. By similar calculations we obtain $P_{(3,1)}(T(1,2) \text{ is odd}) = 27/80$.

$\boldsymbol{(k, \ell) = (2, 2)}$. In (12), $r = 1, 2$, so similar calculations lead to $P_{(2,2)}(T(1,2)$ is odd$) = 9/25$, which already is rather close to $1/e = 0.36788\ldots$.

Remark 5.7: Fix $0 \leq r < d \in \mathbb{N}$. In a similar way it is possible to apply Theorem 4.1 and calculate the probabilities $P_{(k,\ell)}(T(1,2) \equiv r \pmod{d})$. Of course, applying Theorem 3.3, one can calculate such prpbabilities at other cells (i,j). In the following example Corollary 4.2 was applied .

$$P_{(0,3)}(T(1,2)\equiv 0 \pmod 3) = 0.348657,$$
$$P_{(0,3)}(T(1,2)\equiv 1 \pmod 3) = 0.208921,$$
$$P_{(0,3)}(T(1,2)\equiv 2 \pmod 3) = 0.442421$$

while

$$P_{(3,0)}(T(1,2)\equiv 0 \pmod 3) = P_{(3,0)}(T(1,2)=3) = 8/27 = 0.296296,$$
$$P_{(3,0)}(T(1,2)\equiv 1 \pmod 3) = P_{(3,0)}(T(1,2)=4) = 1/27 = 0.037037,$$
$$P_{(3,0)}(T(1,2)\equiv 2 \pmod 3) = P_{(3,0)}(T(1,2)=2) = 2/3 = 0.66667.$$

Compare with the corresponding values for the Plancherel measure $M(0;0;1)$, see [12, (8.4.4)].

$$\mathcal{P}_{M(0;0;1))}(T(1,2)\equiv 0 \pmod 3) = 0.3403$$
$$\mathcal{P}_{M(0;0;1))}(T(1,2)\equiv 1 \pmod 3) = 0.126193$$
$$\mathcal{P}_{M(0;0;1))}(T(1,2)\equiv 2 \pmod 3) = 0.533507$$

6. Precise probabilities, the cases $(2,0)$ and $(1,1)$

Recall That

$$S(k,\ell;n) := |STH(k,\ell;n)|,$$
$$S_{odd}(k,\ell;n) := |\{T_\lambda \in STH(k,\ell;n) \mid T_\lambda(1,2) \text{ is odd}\}|,$$

and that

$$P_{(k,\ell)}(T(1,2) \text{ is odd}) = \lim_{n\to\infty} \frac{S_{odd}(k,\ell;n)}{S(k,\ell;n)}.$$

For few pairs (k,ℓ) with small k and ℓ we now calculate $S_{odd}(k,\ell;n)$ as well as the ratio $S_{odd}(k,\ell;n)/S_(k,\ell;n)$. By sending n to infinity, this yields $P_{(k,\ell)}(T(1,2) \text{ is odd})$.

Note that for $k\le 5$ there are explicit formulas for $S(k,0;n)$, see [4], [5], [13, page 493]. Below we give explicit formulas for $S_{odd}(k,\ell;n)$ in the cases $(k,\ell)\in\{(2,0),(1,1),(2,1)\}$. We also conjecture an explicit formula for $S_{odd}(3,0;n)$.

Some of these formulas are of interest on their own. Moreover, it should be interesting to find bijective proofs for some of these formulas.

6.1. The case $(2,0)$.

Proposition 6.1: $S_{odd}(2,0;n) = S(2,0;n-2)$.

Proof. Let $\lambda = (\lambda_1, \lambda_2) \in H(2,0;n)$, $\lambda_1 \geq 2$, T_λ standard of shape λ, with $T_\lambda(1,2)$ odd. Then $\lambda_2 \geq 1$, that entry must be 3, and its $(2,1)$-entry must be 2. The number of such tableaux is $f^{\lambda \setminus (2,1)}$ = the number of standard tableaux of skew shape $\lambda \setminus (2,1)$. Compare $\lambda \setminus (2,1)$ with the partition $(\lambda_1 - 1, \lambda_2 - 1)$. Assuming that the $(1,1)$-entry in $(\lambda_1 - 1, \lambda_2 - 1)$ is filled, say, with 0, we see that $f^{\lambda \setminus (2,1)} = f^{(\lambda_1 - 1, \lambda_2 - 1)}$. Thus

$$\begin{aligned} S_{odd}(2,0;n) &= \sum_{\substack{\lambda \in H(2,0;n) \\ \lambda_1 \geq 2,\, \lambda_2 \geq 1}} f^{(\lambda_1 - 1, \lambda_2 - 1)} \\ &= \sum_{\mu \in H(2,0;n-2)} f^{\mu} = S(2,0;n-2). \qquad \blacksquare \end{aligned}$$

Note that the above proof is bijective, as it corresponds bijectively to $T_\lambda \in \{T_\lambda \in STH(k,\ell;n) : T_\lambda(1,2) \text{ is odd}\}$ with $T_{\bar\lambda} \in STH(2,0;n-2)$, where for $\lambda = (\lambda_1, \lambda_2)$, $\bar\lambda = (\lambda_1 - 1, \lambda_2 - 1)$.

As usual, $\lfloor \alpha \rfloor$ denotes the integer part of $\alpha \in \mathbb{R}$. Note also that

$$S(2,0;n) = \binom{n}{\lfloor \frac{n}{2} \rfloor} \quad \text{hence } S_{odd}(2,0;n) = \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}, \tag{16}$$

see [10], [13, page 493]. Since

$$\lim_{n \to \infty} \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor} \Big/ \binom{n}{\lfloor \frac{n}{2} \rfloor} = 1/4,$$

it follows that $P_{(2,0)}(T(1,2) \text{ is odd}) = 1/4$, which coinsides with the case $k = 2$ of Example 5.4.

6.2. The case $(k,\ell) = (1,1)$.

Lemma 6.2: (1)

$$S(1,1;n) = \sum_{\lambda \in H(1,1;n)} f^{\lambda} = 2^{n-1}.$$

(2)

$$S_{odd}(1,1;n) = \frac{1}{3}\left[2^{n-1} - \frac{3-(-1)^{n+1}}{2}\right]. \tag{17}$$

It easily follows that

a) $S_{odd}(1,1;2m) = 2S_{odd}(1,1;2m-1)$, *and*

b) $S_{odd}(1,1;2m+1) = 4S_{odd}(1,1;2m-1)+1$.

Proof. Part 1 is well-known, and we prove part 2.

Let $\lambda \in H(1,1;n)$, say $\lambda = (n-q,1^q)$. Assume T_λ is standard with $T_\lambda(1,2) = 2t+1$ its $(1,2)$ entry. Then the first $2t$ entries in the first column of T_λ are $1,2,\ldots,2t$ so necessarily $2t-1 \le q$. Such T_λ is then completely determined if we choose $q-(2t-1)$ elements for the remaining entries of that column; from the $n-(2t+1)$ elements $2t+2, 2t+3, \ldots, n$. The number of such T_λ's is $\binom{n-(2t+1)}{q-(2t-1)} = \binom{n-2t-1}{n-q-2}$. Thus $1 \le q \le n-2$ and also $2 \le 2t \le q+1$, so $1 \le t \le \lfloor (q+1)/2 \rfloor$. Summing on all such t's implies that the number of tableaux T_λ with $\lambda = (n-q,1^q)$ and $T_\lambda(1,2)$ is odd, is $\sum_{t=1}^{\lfloor (q+1)/2\rfloor}\binom{n-2t-1}{n-q-2}$.

Summing on $1 \le q \le n-2$, we obtain that the total number of these tableaux is

$$\begin{aligned} S_{odd}(1,1;n) &= \sum_{q=1}^{n-2}\sum_{t=1}^{\lfloor (q+1)/2\rfloor}\binom{n-2t-1}{n-q-2} = \sum_{q\ge1}\sum_{t\ge1}\binom{n-2t-1}{n-q-2} \\ &= \sum_{t\ge1}\sum_{q\ge1}\binom{n-2t-1}{n-q-2} \\ &= \sum_{t=1}^{\lfloor (n-1)/2\rfloor}\left(\sum_{i\ge1}\binom{n-2t-1}{i}\right) \\ &= \sum_{t=1}^{\lfloor (n-1)/2\rfloor} 2^{n-2t-1}. \end{aligned}$$

By considering the two cases n even and n odd, it easily follows that this last sum equals the r.h.s. of (17), and the proof follows. ∎

This clearly implies that

$$\lim_{n\to\infty} S_{odd}(1,1;n)/S(1,1;n) = 1/3,$$

which co insides with the case $(k,\ell) = (1,1)$ of Example 5.6.

7. The case $(0,2)$ and Catalan and Fine numbers

Following Remark 5.3, we first compare $f^{(2^m)}$ with $h^{(2^m)}$. According to Example 5.5, we should have

$$h^{(2^m)} \approx 4/9 \cdot f^{(2^m)}. \tag{18}$$

While $f^{(2^m)} = f^{(m,m)}$ is the m-th Catalan number $C_m = \frac{1}{m+1}\binom{2m}{m}$, it is shown in Lemma 7.2 below that $h^{(2^m)} = \sum_{t=1}^{\lfloor k/2 \rfloor} 2t\frac{(2k-2t-1)!}{k!(k-2t)!}$, which is the m-th Fine number F_m, namely, $h^{(2^m)} = F_m$, see, for example, [2]. The Fine numbers satisfy the equation $C_m = 2 \cdot F_m + F_{m-1}$, and their asymptotics is $F_m \sim \frac{4^{m+1}}{9\cdot\sqrt{\pi}\cdot m\cdot\sqrt{m}}$. The asymptotics of C_m (easily obtained by the Stirling formula) then imply that, indeed,

$$\lim_{m\to\infty} h^{(2^m)}/f^{(2^m)} = \lim_{m\to\infty} F_m/C_m = 4/9,$$

which agrees with (18).

Turn now to the sums $S_{odd}(0,2;n)$. The following recurrence can be proved for $S_{odd}(0,2;n)$:

Proposition 7.1: 1. *$S_{odd}(0,2;3) = 1$, and for $n \geq 4$, $S(0,2;n)$ satisfies the recurrences.*

2. *$S_{odd}(0,2;2m+1) = 4S_{odd}(0,2;2m-1) + 1 - F_m$, and*
 $S_{odd}(0,2;2m) = 2S_{odd}(0,2;2m-1)$.
3. *Also, $S_{odd}(0,2;2m-1)$ is given explicitly as follows:*

$$S_{odd}(0,2;2m-1) = \frac{1}{9}\left[\binom{2m+1}{m} + 2\cdot F_m - 3\right] \quad \text{and}$$

$$S_{odd}(0,2;2m) = \frac{2}{9}\left[\binom{2m+1}{m} + 2\cdot F_m - 3\right].$$

Some details are given in the Appendix. Here we prove

Lemma 7.2: *$h^{(2^m)} = F_m$, the m-th Fine number.*

Proof. Let $T = T_{(2^m)}$ be a standard tableau of shape (2^m) and with $T(1,2) = 2t+1$. Then the first $2t$ entries of its first column are $1,\ldots,2t$. Delete the cells containing $1,\ldots,2t+1$ and obtain a standard tableau T^- of a certain skew shape. Conjugate that skew shape, then rotate it by $180°$, and finally reverse the order among its entries, so it becomes standard. It follows that the number of such T's is $f^{(2^{m-2t},2t-1)} = f^{(m-1,m-2t)}$. Summing on $1 \leq t \leq [m/2]$ and applying, say, the hook-formula for calculating f^λ, it follows that

$$h^{(2^m)} = \sum_{t=1}^{\lfloor m/2 \rfloor} 2t \frac{(2m-2t-1)!}{m!(m-2t)!} = \sum_t \frac{t}{m-t} \cdot \binom{3m-2t}{m}. \tag{19}$$

The proof now follows since it is known that the $r.h.s = F_m$, see, for example, [2, page 251]. ∎

Remark 7.3: It is known that F_m is the number of standard tableaux of shape (m, m) without a column of the form

$$\begin{array}{|c|}\hline a \\ \hline a' \\ \hline\end{array}$$

where $a' = a + 1$. Thus, there is a bijection between such tableaux and the tableaux T of shape (2^m) with $T(1, 2)$ odd [2]. It should be interesting to find an explicit such bijection.

8. A closed formula for $S(2, 1; n)$

Theorem 8.1: *Recall that $S(2, 1; n) = \sum_{\lambda \in H(2,1;n)} f^\lambda$ and let $n \geq 2$, then*

$$\begin{aligned}
&\sum_{\lambda \in H(2,1;n)} f^\lambda \\
&\quad = \sum_{\lambda \in H(1,2;n)} f^\lambda \\
&\quad = \frac{1}{4}\Bigg(\sum_{r=0}^{n-1} \binom{n-r}{\lfloor \frac{n-r}{2} \rfloor} \binom{n}{r} \\
&\qquad\quad + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{n!}{k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k)} \Bigg) + 1.
\end{aligned}$$

The proof applies Pieri's rule for the "outer" product $\chi^\mu \hat{\otimes} \chi^{(1^n)}$, see [7, I,(5.16),(5.17)]. We shall need the following three lemmas.

Lemma 8.2: *Let $\eta(n)$ be the following S_n-character*

$$\eta(n)=\sum_{r=0}^{n-1}\Big(\sum_{\mu\in H(2,0;n-r)}\chi^{\mu}\Big)\hat{\otimes}\chi^{(1^r)},\quad \textit{then } \eta(n)=\sum_{\lambda\in H(2,1;n)} b(\lambda)\chi^{\lambda},$$

where $b((n))=b((1^n))=2$, $b((k+1,k+1,1^{n-2k-2}))=3$ if $\lambda_1=\lambda_2=k+1\geq 2$, namely, if $1\leq k\leq \lfloor \frac{n}{2}\rfloor -1$, and $b(\lambda)=4$ in all other cases.

Proof. By Pieri's rule it clearly follows that $\eta(n)$ is supported on the $(2,1)$ hook. Given $\lambda\in H(2,1;n)$, calculate the number of partitions $\mu\in H(2,0)$ ($|\mu|\leq n$) such that χ^{λ} appears in $\chi^{\mu}\hat{\otimes}\chi^{(1^r)}$ (of course, with multiplicity 1).

1. The case $\lambda_1>\lambda_2\geq 1$. There are four possible μ's here:
$$(\lambda_1-1,\lambda_2-1),(\lambda_1-1,\lambda_2),(\lambda_1,\lambda_2-1),\ \text{and}\ (\lambda_1,\lambda_2).$$
2. The case $\lambda_1=\lambda_2\geq 2$. There are three possible μ's in this case:
$$(\lambda_1-1,\lambda_2-1),(\lambda_1,\lambda_2-1),\ \text{and}\ (\lambda_1,\lambda_2).$$
3. The case $\lambda_1=1$, hence $\lambda=(1^n)$. The two possible μ's in this case are: (1) and (1^2).

The proof of the lemma is complete. ∎

Lemma 8.3: *The degree of the character $\eta(n)$ in Lemma 8.2 is the following sum $s_1(n)$:*

$$s_1(n)=\deg\left(\sum_{r=0}^{n-1}\Big(\sum_{\mu\in H(2,0;n)}\chi^{\mu}\Big)\hat{\otimes}\chi^{(1^r)}\right)=\sum_{r=0}^{n-1}\binom{n-r}{\lfloor\frac{n-r}{2}\rfloor}\binom{n}{r}.$$

Proof. First, it is well-known that for any two partitions μ and ν,

$$\deg(\chi^{\mu}\hat{\otimes}\chi^{\nu})=\deg(\chi^{\mu})\deg(\chi^{\nu})\binom{|\mu|+|\nu|}{|\mu|}.$$

The second fact used in the proof is that

$$\sum_{\mu\in H(2,0;m)} f^{\mu}=\binom{m}{\lfloor\frac{m}{2}\rfloor},$$

[10, Section 4], [13, Exercise 7.16]. The proof of the lemma now easily follows. ∎

The hook formula for the f^{λ}'s easily implies the following lemma.

Lemma 8.4: *Let*

$$s_2(n) := \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor -1} \frac{n!}{k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k)}$$

be the second sum in Theorem 8.1, and let $\rho(n)$ denote the following S_n-character:

$$\rho(n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor -1} \chi^{(k+1,k+1,1^{(n-2k-2)})}.$$

Then $\deg(\rho(n)) = s_2(n)$.

The proof of Theorem 8.1.

Proof. Form the S_n–character $\eta(n) + \rho(n) + 2\chi^{(n)} + 2\chi^{(1^n)}$. By Lemma 8.2,

$$\eta(n) + \rho(n) + 2\chi^{(n)} + 2\chi^{(1^n)} = 4 \sum_{\lambda \in H(2,1;n)} \chi^{\lambda}.$$

The proof now follows from the previous lemmas by taking the degrees of these characters. ∎

9. A closed formula for $S_{odd}(2,1;n)$

Lemma 9.1: *Let $n \geq 3$, then*

$$S_{odd}(2,1;n) = \sum_{q=0}^{n-3} \binom{n-q-2}{\lfloor (n-q-2)/2 \rfloor} \sum_{t=1}^{\lfloor (q+2)/2 \rfloor} \binom{n-2t-1}{n-q-3}.$$

Proof. Let $\lambda \in H(2,1;n)$, T_λ standard with $T_\lambda(1,2)$ odd, so $\lambda_1 \geq 2,\ \lambda_2 \geq 1$. Denote $\lambda = (\lambda_1, \lambda_2, 1^q)$ (so $(\lambda_1 + \lambda_2 = n-q)$, and let $T_\lambda(1,2) = 2t+1$, where $t \geq 1$. Then $1, 2, \ldots, 2t$ occupy the first $2t$ entries of the first column, leaving $q+2-2t$ entries free. Choose these $q+2-2t$ entries from the remaining $n-(2t+1)$ numbers $2t+2, \ldots, n$ to fill the first column (in a unique standard way). This can be done in $\binom{n-(2t+1)}{q+2-2t} = \binom{n-2t-1}{n-q-3}$ ways.

The remaining entries occupy the skew partition $(\lambda_1 - 1, \lambda_2 - 1)/(1)$ and form a standard skew tableau. Filling the missing top-left-cell of $(\lambda_1 - 1, \lambda_2 - 1)/(1)$ with 0 implies there are $f^{(\lambda_1 - 1, \lambda_2 - 1)}$ such standard fillings. Since $\mu = (\lambda_1 - 1, \lambda_2 - 1) \vdash n-q-2$ is an arbitrary two parts partition of $n-q-2$, by (16) the number of such standard tableaux T_μ is $\binom{n-q-2}{\lfloor (n-q-2)/2 \rfloor}$. Thus, with such $0 \le q \le n-3$ and $3 \le 2t+1 \le q+3$ (so $1 \le t \le \lfloor (q+2)/2 \rfloor$), the total number of such tableaux is

$$\sum_{q=0}^{n-3} \sum_{t=1}^{\lfloor (q+2)/2 \rfloor} \binom{n-q-2}{\lfloor (n-q-2)/2 \rfloor} \binom{n-2t-1}{n-q-3} = \sum_{q=0}^{n-3} \binom{n-q-2}{\lfloor (n-q-2)/2 \rfloor} \sum_{t=1}^{\lfloor (q+2)/2 \rfloor} \binom{n-2t-1}{n-q-3}. \quad \blacksquare$$

We have the following surprising identity.

Proposition 9.2: *$S(2,1;n) = 1 + S_{odd}(2,1;n+1)$, namely*

$$\sum_{\lambda \in H(2,1;n)} f^\lambda = 1 + \sum_{\lambda \in H(2,1;n+1)} h^\lambda$$

By Theorem 8.1 and by Lemma 9.1, this identity is equivalent to the following binomial identity

$$(20) \quad \frac{1}{4}\Bigg(\sum_{r=0}^{n-1} \binom{n-r}{\lfloor \frac{n-r}{2} \rfloor} \binom{n}{r} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{n!}{k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k)} \Bigg) = \sum_{q=0}^{n-2} \binom{n-q-1}{[(n-q-1)/2]} \sum_{j=1}^{\lfloor (q+2)/2 \rfloor} \binom{n-2j}{n-q-2}.$$

Proof. We are thankful to D. Zeilberger who verified this identity by the WZ method. ∎

Remark 9.3: If we add 1 to both sides of (20), then Theorem 8.1 gives a combinatorial interpretation to the l.h.s. while Lemma 9.1 gives such an interpretation to the r.h.s. This probably hints at a bijective proof of Proposition 9.2.

10. The cases $S_{odd}(3,0;n)$ and $S_{odd}(1,2;n)$

We begin by recalling the following formula, see [10], [13]:

$$S(3,0;n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(k+1)!(n-2k)!}.$$

By direct calculations, for $n = 3, 4, \ldots, 14$ we have:

$$S_{odd}(3,0;n) = 1, 3, 7, 17, 42, 106, 272, 708, 1865, 4963, 13323, 36037$$
$$S(3,0;n) = 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634$$

This indicates of the following

Conjecture 10.1: $S_{odd}(3,0;n) = S(3,0;n-1) - S(3,0;n-3)$.

It should be interesting to find a bijective proof to that conjecture, namely, a bijection between the corresponding tableaux. Note that since $S_3(n) \sim a \cdot n^b \cdot 3^n$, where a, b are constants, the conjecture would imply that

$$\lim_{n\to\infty} \frac{S_{odd}(3,0;n)}{S(3,0;n)} = \frac{3^{n-1} - 3^{n-3}}{3^n} = \frac{8}{27},$$

agreeing with the case $k = 3$ of Example 5.4.

Remark 10.2: Conjecture 10.1 has just been proved by Shalosh B. Ekhad and D. Zeilberger [3].

In the case $S_{odd}(1,2;n)$, by direct calculations for $S_{odd}(1,2;n)$ and by Theorem 8.1 we have the following values ($n = 1, 2, 3, \ldots$)

$$S_{odd}(1,2;n): \ 0, 0, 1, 3, 9, 25, 71, 201, 573, 1639, 4708, 13568, 39218, 113646$$
$$S(1,2;n): 1, 2, 4, 10, 26, 71, 197, 554, 1570, 4477, 12827, 36895, 106471, 308114$$

Example 10.3: Let $\lambda = (m, m, m)$ and show that as m goes to infinity, h^λ / f^λ indeed goes to $8/27$. Let T be a standard tableau of shape $\lambda = (m, m, m)$ and with $T(1, 2)$ odd. Then necessarily $T(1, 2) = 3$ and $T(2, 1) = 2$. Rotating the skew tableau $T/(2, 1)$ by 180°, it follows that $h^{(m,m,m)} = f^{(m,m-1,m-2)}$ by, say, the hook-formula,

$$\frac{h^{(m,m,m)}}{f^{(m,m,m)}} = 8 \frac{(m-1)m(m+1)}{(3m-2)(3m-1)3m} \longrightarrow \frac{8}{27}.$$

By similar arguments

$$\frac{h^{(3^m)}}{f^{(3^m)}} = 4 \sum_{t=1}^{\lfloor m/2 \rfloor} t(t+1) \frac{(m-2t+1)(m-2t+2)\cdots(m+1)}{(3m-2t)(3m-2t+1)\cdots(3m)}.$$

Remark 3.7 and the case $\ell = 3$ of Example 5.5 imply that as $m \to \infty$,

$$h^{(3^m)}/f^{(3^m)} \longrightarrow 27/64,$$

and therefore

$$4\sum_{t=1}^{[m/2]} t(t+1)\frac{(m-2t+1)(m-2t+2)\cdots(m+1)}{(3m-2t)(3m-2t+1)\cdots(3m)} \longrightarrow \frac{27}{64}.$$

Remark 10.4: We believe there should be bijective proofs to several of the cases discussed in this paper.

1. The Fine numbers count
   a) The number of standard tableaux T of shape (m,m) with $T(1,2)$ odd, and
   b) The number of standard tableaux T of shape (m,m) without a column of the form

$$\begin{array}{|c|}\hline a \\ \hline a' \\ \hline\end{array}$$

   where $a' = a+1$.

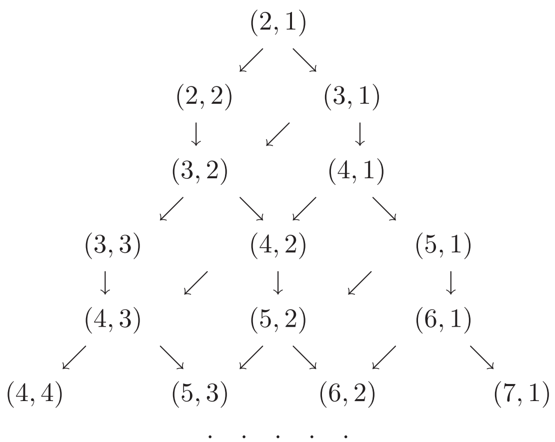
   Give a bijection between these tableaux.
2. By Proposition 9.2, $S(2,1;n) = S_{odd}(2,1;n) + 1$. Find a bijection between the corresponding sets of standard tableaux.
3. Find a bijective proof for Conjecture 10.1.

11. Appendix: $S_{odd}(0,2;n)$ re-visited

11.1. Catalan and Fine triangles. We begin with the following observation (see Definition 5.2).

(1) $h^{(2,1^{n-2})} = h^{(n-1,1)'} = \lfloor \frac{n-1}{2} \rfloor$, $n \ge 3$, and
(2) If $\lambda_2 \ge 2$, then $h^{(\lambda_1,\lambda_2)'} = h^{(\lambda_1-1,\lambda_2)'} + h^{(\lambda_1,\lambda_2-1)'}$. Here $h^{(\lambda_1-1,\lambda_2)'} = 0$ if $\lambda_1 = \lambda_2$.

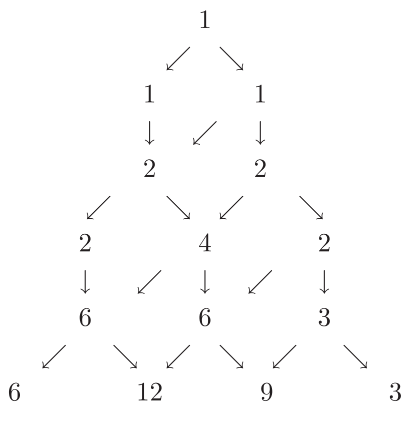
This leads to the following infinite triangle: construct the part of the Young graph of the proper-two-parts partitions:



The proper two parts partitions.

where (λ_1, λ_2) is connected with $(\lambda_1 + 1, \lambda_2)$ and with $(\lambda_1, \lambda_2 + 1)$ (provided $\lambda_1 \geq \lambda_2 + 1$).

With each partition λ, write its corresponding $h^{\lambda'}$ (the partitions (n) are deleted since $h^{(n)'} = h^{(1^n)} = 0$). We obtain the following "Fine-triangle"



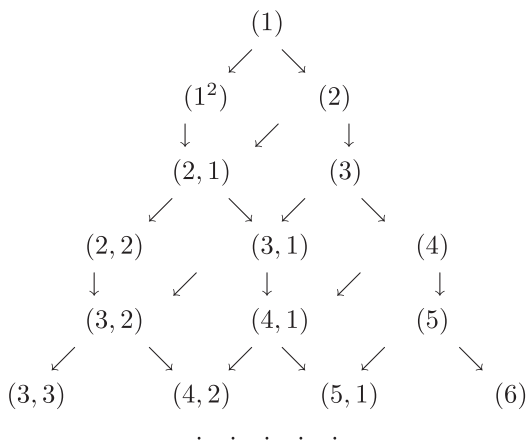
A Fine triangle.

Note that the rightmost numbers are the multiplicities $h^{(n-1,1)'} = \lfloor \frac{n-1}{2} \rfloor$, but all other numbers are obtained by adding the incoming multiplicities. Clearly, the rightmost numbers are the natural numbers repeated twice: $1, 1, 2, 2, 3, 3, 4, 4, \ldots$. The leftmost numbers also repeat twice. After deleting repetitions, obtain the sequence $1, 2, 6, 18, 57, 186, \ldots$ which are the Fine numbers. The row-sums in the Fine triangle are the numbers $S_{odd}(0,2;n)$: $1,2,4,8,15,30,\ldots$. From the above Fine-triangle it follows that

1. $S_{odd}(0,2;2m) = 2S_{odd}(0,2;2m-1)$, and
2. $S_{odd}(0,2;2m+1) = 2S_{odd}(0,2;2m) + 1 - F_m$,

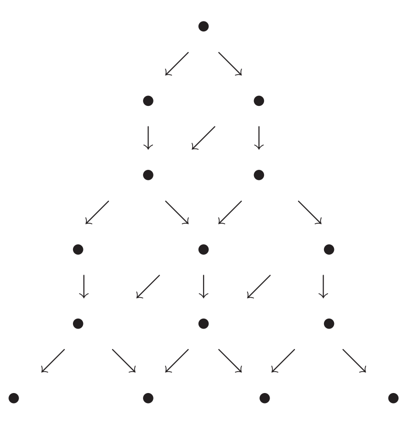
see Proposition 7.1. Note that in the Fine triangle, the multiplicities on the q-th diagonal from North–East to South–West are the numbers $h^{(n,q-1)'}$.

11.2. A Catalan triangle. Now write all the two-parts partitions, starting with (1):



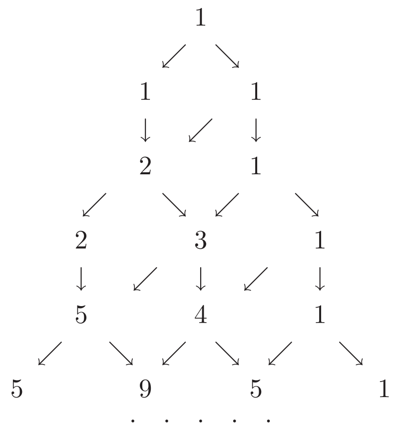
The two parts partitions.

Ignoring multiplicities, this is the same “bullet” graph as in the Fine-case:



The Bullet graph.

At each partition λ now write its corresponding degree $f^{\lambda'} = f^{\lambda}$, and obtain the following "Catalan triangle":



A Catalan Triangle.

In the Catalan triangle, all rightmost numbers are 1 and, like in the Fine graph, all numbers are obtained by adding the incoming multiplicities. The leftmost numbers are the Catalan numbers C_n. The sum of the numbers of the r-th raw are the numbers $S(2,0;r) = \binom{r}{\lfloor \frac{r}{2} \rfloor}$. The numbers on the q-th diagonal from North–East to South–West are the numbers $f^{(n,q-1)}$.

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